Published by Institute of Physics Publishing for SISSA/ISAS

Received: *November 18, 2004*
Revised: *January 10, 2005*
Accepted: *January 17, 2005*

Supersymmetric integrable scattering theories with unstable particles

Andreas Fring

Centre for Mathematical Science, City University
Northampton Square, London EC1V 0HB, UK
E-mail: A.Fring@city.ac.uk

Abstract: We propose scattering matrices for $N = 1$ supersymmetric integrable quantum field theories in 1+1 dimensions which involve unstable particles in their spectra. By means of the thermodynamic Bethe ansatz we analyze the ultraviolet behaviour of some of these theories and identify the effective Virasoro central charge of the underlying conformal field theories.

Keywords: Field Theories in Lower Dimensions, Nonperturbative Effects, Integrable Field Theories.

Contents

1. Introduction **1**

2. $N = 1$ **supersymmetry, generalities** **2**

3. The bootstrap construction **3**
3.1 Constraints from supersymmetry 4
3.2 Constraints from the Yang-Baxter equations 5
3.3 Constraints from hermitian analyticity, unitarity and crossing 5
3.4 Constraints from the boundstate bootstrap equations 7

4. Implementing unstable particles **8**

5. The ultraviolet limit, a TBA analysis **9**
5.1 The $\mathbf{A}_{2\ell}^{(2)}|\tilde{\mathbf{g}}$-theories 11
5.2 The $N = 1$ supersymmetric $\mathbf{A}_{2\ell}^{(2)}|\tilde{\mathbf{g}}$-theories 11

6. Conclusion **12**

1. Introduction

Supersymmetry is a natural concept in particle physics changing bosons into fermions and vice versa in a well controlled manner. The original idea traces back over thirty years and its discovery is attributed to Golfand and Likhtman [1]. Initially the understanding was mainly developed in the context of string theory [2, 3, 4], where supersymmetry plays the important role of a two-dimensional symmetry of the world sheet. Thereafter it became a more widely, albeit not universally, accepted principle when the two dimensional symmetry was generalized to four dimensions for the Wess-Zumino model [5]. Whereas most symmetries lead to trivial scattering theories, as a consequence of the Coleman-Mandula theorem [6], supersymmetry is a very special one in the sense that it can be present and still does not prevent the theory to be non-trivial. This holds even in higher dimensions. Hence, supersymmetry is regarded as one of the concepts worthwhile studying in 1+1 dimensions as it might even have a direct bearing on higher dimensional theories.

The first scattering matrices for N=1 supersymmetric integrable quantum field theories in 1+1 dimensions were constructed by Witten and Shankar in the late seventies [7] for theories with degenerate mass spectra. In a systematic manner these results were generalized by Schoutens twelve years later to theories with non-degenerate mass spectra [8], see also [9]–[14]. S-matrices for some specific cases of $N = 2$ supersymmetric theories

JHEP01(2005)030

were constructed by Köberle and Kurak [15] in the late eighties and thereafter generalized [16, 17, 18]. Up to now all the constructed S-matrices invariant under supersymmetry involve exclusively stable particles in their spectra.

Even though it has been commented upon the occurrence of unstable particles in integrable quantum field theories in 1+1 dimensions as early as the late seventies [19], only recently such type of theories have been investigated in more detail. There is a particularly interesting class of models, the homogeneous sine-Gordon models, for which the spectrum of unstable particles can be associated directly to a classical lagrangian. It is these model we deal with in this manuscript. For such type of theories scattering matrices have been constructed [20, 21, 22], their ultraviolet behaviour has been analyzed by means of the thermodynamic Bethe ansatz [23]–[27] and also form factors have been constructed which were used to compute correlation functions needed in various quantities [28]–[31].

The purpose of this note is to propose and analyze the class of scattering matrices associated to the homogeneous sine-Gordon models which are invariant under supersymmetry transformations and contain besides stable particles also unstable particles in their spectra.

Resonances can also be introduced by an analytic continuation of the effective coupling constant [32]–[42]. We shall only briefly comment on these models as their supersymmetric version is trivial to obtain, albeit it would be interesting to investigate their renormalization group flows, whose interpretation is less direct as for the homogeneous sine-Gordon models.

The manuscript is organized as follows: in the next section we briefly recall some of the main features of $N = 1$ supersymmetry relevant for the development of a scattering theory. In section 3 we present the bootstrap construction for supersymmetric theories and in section 4 a proposal which implements in addition unstable particles into such type of theories. In section 5 we investigate the ultraviolet limit for some of the proposed S-matrices. The conclusions are stated in section 6.

2. $N = 1$ supersymmetry, generalities

We commence by fixing the notation and briefly recall the key features of $N = 1$ supersymmetry which will be relevant below. The prerequisite for a supersymmetric theory is the existence of two conserved supercharges $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ together with a fermion parity operator $\hat{\mathcal{Q}}$. The $N = 1$ superalgebra obeyed by these charges reads in general

$$\{\hat{\mathcal{Q}}, \mathcal{Q}\} = \{\hat{\mathcal{Q}}, \bar{\mathcal{Q}}\} = 0\,, \quad \mathcal{Q}^2 = \mathcal{P}_1\,, \quad \bar{\mathcal{Q}}^2 = \mathcal{P}_{-1}\,, \quad \hat{\mathcal{Q}}^2 = \mathbb{I} \quad \text{and} \quad \{\mathcal{Q}, \bar{\mathcal{Q}}\} = T\,, \qquad (2.1)$$

where $\mathcal{P}_{\pm 1}$ are charges of Lorentz spin ± 1 and T is the topological charge operator. Here we restrict ourselves to the case $T = 0$ (see [8] for some examples with $T \neq 0$). Next we note that in a massive $N = 1$ supersymmetric theory one can arrange all particles in multiplets (b_i, f_i) with internal quantum numbers $1 \leq i \leq \ell$ containing a boson b_i and a fermion f_i with equal masses $m_{b_i} = m_{f_i} = m_i$. An asymptotic boson or fermion is characterized by a creation operator $Z_{\mu_j}(\theta)$ depending on the rapidity θ, which parameterizes the momenta as $p^0 = m \cosh\theta$, $p^1 = m \sinh\theta$. In agreement with (2.1), one [7, 8] can specify the action of the charges on asymptotic n-particle states. In particular, on a one-particle asymptotic state they act as

$$\mathcal{Q}\left|Z_{\mu_j}(\theta)\right\rangle_{\text{in/out}} = \sqrt{m_j}e^{\theta/2}\left|Z_{\hat{\mu}_j}(\theta)\right\rangle_{\text{in/out}} \qquad \text{for} \quad \mu = b, f; \quad 1 \leq i \leq \ell\,, \tag{2.2}$$

$$\bar{\mathcal{Q}}\left|Z_{\mu_j}(\theta)\right\rangle_{\text{in/out}} = i(-1)^{F_{\mu_j}}\sqrt{m_j}e^{\theta/2}\left|Z_{\hat{\mu}_j}(\theta)\right\rangle_{\text{in/out}} \qquad \text{for} \quad \mu = b, f; \quad 1 \leq i \leq \ell\,, \tag{2.3}$$

$$\hat{\mathcal{Q}}\left|Z_{\mu_j}(\theta)\right\rangle_{\text{in/out}} = (-1)^{F_{\mu_j}}\left|Z_{\mu_j}(\theta)\right\rangle_{\text{in/out}} \qquad \text{for} \quad \mu = b, f; \quad 1 \leq i \leq \ell\,, \tag{2.4}$$

where we defined

$$\hat{\mu} = \begin{cases} b & \text{for } \mu = f \\ f & \text{for } \mu = b \end{cases} \qquad \text{and} \qquad F_\mu = \begin{cases} 1 & \text{for } \mu = f \\ 0 & \text{for } \mu = b\,. \end{cases} \tag{2.5}$$

For the generalization to an action on n-particle asymptotic states one defines

$$\mathcal{Q}^{(n)} = \sum_{k=1}^{n}\left(\bigotimes_{l=1}^{k-1}\hat{\mathcal{Q}}_l\right)\mathcal{Q}_k\,, \qquad \bar{\mathcal{Q}}^{(n)} = \sum_{k=1}^{n}\left(\bigotimes_{l=1}^{k-1}\hat{\mathcal{Q}}_l\right)\bar{\mathcal{Q}}_k\,, \qquad \hat{\mathcal{Q}}^{(n)} = \bigotimes_{k=1}^{n}\hat{\mathcal{Q}}_k\,. \tag{2.6}$$

As we want to construct two-particle scattering amplitudes, we require in particular the action of these charges on two-particle states. From the above definitions one obtains

$$\begin{aligned}
\hat{\mathcal{Q}}^{(2)}\left|Z_{\mu_j}(\theta)Z_{\nu_k}(\theta')\right\rangle_{\text{in/out}} &= e^{i\pi(F_{\mu_j}+F_{\nu_k})}\left|Z_{\mu_j}(\theta)Z_{\nu_k}(\theta')\right\rangle_{\text{in/out}} \\
\mathcal{Q}^{(2)}\left|Z_{\mu_j}(\theta)Z_{\nu_k}(\theta')\right\rangle_{\text{in/out}} &= \sqrt{m_j}e^{\frac{\theta}{2}}\left|Z_{\hat{\mu}_j}(\theta)Z_{\nu_k}(\theta')\right\rangle_{\text{in/out}} + \sqrt{m_k}e^{\frac{\theta'}{2}}\left|Z_{\mu_j}(\theta)Z_{\hat{\nu}_k}(\theta')\right\rangle_{\text{in/out}} \\
\bar{\mathcal{Q}}^{(2)}\left|Z_{\mu_j}(\theta)Z_{\nu_k}(\theta')\right\rangle_{\text{in/out}} &= i\sqrt{m_j}e^{\frac{\theta}{2}+i\pi F_{\mu_j}}\left|Z_{\hat{\mu}_j}(\theta)Z_{\nu_k}(\theta')\right\rangle_{\text{in/out}} + \\
&\quad + i\sqrt{m_k}e^{\frac{\theta'}{2}+i\pi(F_{\mu_j}+F_{\nu_k})}\left|Z_{\mu_j}(\theta)Z_{\hat{\nu}_k}(\theta')\right\rangle_{\text{in/out}}\,.
\end{aligned} \tag{2.7}$$

This is sufficient information to determine the consequences on the scattering theory of an integrable quantum field theory when demanding it to be supersymmetric.

3. The bootstrap construction

Next recall briefly the key steps of the bootstrap construction carried out by Schoutens [8] with some minor differences. As a general structure for S one assumes usually [7, 8] the factorization into a purely bosonic part $\hat{S}$ and a factor $\check{S}$ which incorporates the boson-fermion mixing. Hence

$$\left|Z_{\alpha_i}(\theta)Z_{\beta_j}(\theta')\right\rangle_{\text{in}} = \sum_{k,l=1}^{\ell}\sum_{\gamma,\delta=b,f}\hat{S}_{ij}^{kl}(\theta-\theta')\check{S}_{\alpha_i\beta_j}^{\gamma_k\delta_l}(\theta-\theta')\left|Z_{\gamma_k}(\theta')Z_{\delta_l}(\theta)\right\rangle_{\text{out}}\,. \tag{3.1}$$

We use here the notation that the Latin indices are the internal quantum numbers relating to the particle type and the Greek indices distinguish fermions f from bosons b. This means by construction $\hat{S}$ commutes trivially with all supercharges $\mathcal{Q}$, $\bar{\mathcal{Q}}$ and $\hat{\mathcal{Q}}$, whereas the requirement of invariance under supersymmetry constrains only the boson-fermion mixing factor $\check{S}$. It seems there is no compelling reason for demanding the factorization (3.1) and one could envisage more general constructions, but we follow here [7, 8] and take $S = \hat{S}\check{S}$ as a working hypothesis. One can now invoke consecutively the consistency equations from the bootstrap program [43]–[46] in order to determine the precise form of S.

3.1 Constraints from supersymmetry

Following the most systematic treatment [8], one can first of all analyze the constraints resulting from the requirement that the theory should be supersymmetric, which means the supercharges should commute with the scattering matrix. Paying attention to the fact that S intertwines the asymptotic states it follows with (2.7), that the boson-fermion mixing matrix $\check{S}$ has to obey

$$\left[\hat{Q}\otimes\hat{Q}\right]\check{S}(\theta_{12}) \;=\; \check{S}(\theta_{12})\left[\hat{Q}\otimes\hat{Q}\right] \tag{3.2}$$

$$\left[m_k^{\frac{1}{2}}e^{\frac{\theta_2}{2}}Q\otimes\mathbb{I}+m_j^{\frac{1}{2}}e^{\frac{\theta_1}{2}}\hat{Q}\otimes Q\right]\check{S}(\theta_{12}) \;=\; \check{S}(\theta_{12})\left[m_j^{\frac{1}{2}}e^{\frac{\theta_1}{2}}Q\otimes\mathbb{I}+m_k^{\frac{1}{2}}e^{\frac{\theta_2}{2}}\hat{Q}\otimes Q\right] \tag{3.3}$$

$$\left[m_k^{\frac{1}{2}}e^{\frac{-\theta_2}{2}}\bar{Q}\otimes\mathbb{I}+m_j^{\frac{1}{2}}e^{\frac{-\theta_1}{2}}\hat{Q}\otimes\bar{Q}\right]\check{S}(\theta_{12}) \;=\; \check{S}(\theta_{12})\left[m_j^{\frac{1}{2}}e^{\frac{-\theta_1}{2}}\bar{Q}\otimes\mathbb{I}+m_k^{\frac{1}{2}}e^{\frac{-\theta_2}{2}}\hat{Q}\otimes\bar{Q}\right]. \tag{3.4}$$

As usual we abbreviated the rapidity difference $\theta_{12} := \theta_1 - \theta_2$. With the explicit realization [7, 8] for the $N = 1$-superalgebra

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \bar{Q} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \text{and} \qquad \hat{Q} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.5}$$

one can solve (3.2)–(3.4). First one notices that (3.2) implies that the fermion parity has to be the same in the in- and out-state such that only the eight processes $bb \rightarrow (bb, ff)$, $bf \rightarrow (bf, fb)$, $fb \rightarrow (bf, fb)$ and $ff \rightarrow (bb, ff)$ can possibly be non-vanishing. Invoking then also the relations (3.3)–(3.4) fixes the S-matrix up to two entries. Instead of leaving two amplitudes unknown at this stage, it is convenient to introduce [8] two unknown functions $f_{ij}(\theta)$ and $g_{ij}(\theta)$. These functions carry only two indices when one makes the further assumption that the bosonic S-matrix describes a theory with a non-degenerate mass spectrum such that backscattering is absent and $\hat{S}$ is diagonal in the sense of $\hat{S}_{ij}^{kl}(\theta) = \delta_i^l \delta_j^k \hat{S}_{ij}(\theta)$. Even though, when taking the bosonic factor of S to be diagonal, the mass degeneracy between bosons and fermions of the same type forces $\check{S}$ to be of the form

$$\check{S}_{ij}(\theta) = \begin{pmatrix} \check{S}_{b_i b_j}^{b_j b_i}(\theta) & 0 & 0 & \check{S}_{b_i b_j}^{f_j f_i}(\theta) \\ 0 & \check{S}_{b_i f_j}^{b_j f_i}(\theta) & \check{S}_{b_i f_j}^{f_j b_i}(\theta) & 0 \\ 0 & \check{S}_{f_i b_j}^{b_j f_i}(\theta) & \check{S}_{f_i b_j}^{f_j b_i}(\theta) & 0 \\ \check{S}_{f_i f_j}^{b_j b_i}(\theta) & 0 & 0 & \check{S}_{f_i f_j}^{f_j f_i}(\theta) \end{pmatrix}, \qquad \text{for } \; 1 \leq i,j \leq \ell\,. \tag{3.6}$$

To avoid the occurrence of additional phase factors one can include them directly into the asymptotic states and change $Z_{f_j}(\theta) \rightarrow \exp(-i\pi/4) Z_{f_j}(\theta)$. In this new basis Schoutens found [8] as solutions to (3.2)–(3.4)

$$\check{S}_{ij}(\theta) = \frac{2f_{ij}(\theta)}{\rho_{ij}^{+} + \cosh\frac{\theta}{2}} \begin{pmatrix} \rho_{ij}^{+} & 0 & 0 & -i\sinh\frac{\theta}{2} \\ 0 & \cosh\frac{\theta}{2} & -\rho_{ij}^{-} & 0 \\ 0 & \rho_{ij}^{-} & \cosh\frac{\theta}{2} & 0 \\ -i\sinh\frac{\theta}{2} & 0 & 0 & \rho_{ij}^{+} \end{pmatrix} + g_{ij}(\theta) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.7}$$

JHEP01(2005)030

with $\rho_{ij}^{\pm} = \left[(m_i/m_j)^{1/2} \pm (m_j/m_i)^{1/2}\right]/2$. One observes that the requirement of supersymmetry invariance puts severe constraints on the general structure of the S-matrix, albeit it does not fix it entirely. Thus leaving fortunately enough freedom to incorporate also other necessary features.

3.2 Constraints from the Yang-Baxter equations

Next we invoke the equations which are the consequence of the factorizability of the n-particle S-matrix into two particle scattering amplitudes. Since we have mass degeneracy between bosons and fermions of the same type backscattering is possible and the Yang-Baxter equations [47, 48]

$$\sum_{\kappa_1,\kappa_2,\kappa_3} S_{\mu_1\mu_2}^{\kappa_1\kappa_2}(\theta_{12}) S_{\kappa_2\mu_3}^{\kappa_3\nu_1}(\theta_{13}) S_{\kappa_1\kappa_3}^{\nu_3\nu_2}(\theta_{23}) = \sum_{\kappa_1,\kappa_2,\kappa_3} S_{\mu_2\mu_3}^{\kappa_1\kappa_2}(\theta_{23}) S_{\mu_1\kappa_1}^{\nu_3\kappa_3}(\theta_{13}) S_{\kappa_3\kappa_2}^{\nu_2\nu_1}(\theta_{12}) \qquad (3.8)$$

will impose further non-trivial constraints on S. It was noted in [8], that in order to satisfy (3.8) with (3.7) one can fix the ratio between the functions $f_{ij}(\theta)$ and $g_{ij}(\theta)$ up to an unknown constant κ

$$f_{ij}(\theta) = \frac{\kappa\sqrt{m_i m_j}}{2} \left(\frac{\rho_{ij}^{+} + \cosh\frac{\theta}{2}}{\sinh\theta} \right) g_{ij}(\theta)\,, \qquad (3.9)$$

such that

$$\check{S}_{ij}(\theta) = g_{ij}(\theta) \left[\frac{\kappa\sqrt{m_i m_j}}{\sinh\theta} \begin{pmatrix} \rho_{ij}^{+} & 0 & 0 & -i\sinh\frac{\theta}{2} \\ 0 & \cosh\frac{\theta}{2} & -\rho_{ij}^{-} & 0 \\ 0 & \rho_{ij}^{-} & \cosh\frac{\theta}{2} & 0 \\ -i\sinh\frac{\theta}{2} & 0 & 0 & \rho_{ij}^{+} \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right]. \qquad (3.10)$$

If we were dealing with a lattice model we would have already solved the problem to find a consistent supersymmetric R-matrix in that case. However, aiming at the description of a quantum field theory we also have to incorporate all the analytic properties.

3.3 Constraints from hermitian analyticity, unitarity and crossing

A scattering matrix belonging to a proper quantum field theory has to be hermitian analyticity [49, 50], unitarity and crossing invariant [51, 43, 44, 45]

$$S_{ij}^{kl}(\theta_{ij}) = \left[S_{kl}^{ij}(-\theta_{ij}^{*})\right]^{*}, \qquad \sum_{kl} S_{ij}^{kl}(\theta)\left[S_{nm}^{kl}(\theta)\right]^{*} = \delta_{\rm in}\delta_{jm}\,, \qquad S_{ij}^{kl}(\theta_{ij}) = S_{\bar{k}i}^{l\bar{j}}(i\pi - \theta_{ij})\,. \qquad (3.11)$$

It is easy to convince oneself that hermitian analyticity and crossing are satisfied when $\kappa \in \mathbb{R}$ and in addition

$$g_{kj}(\theta) = g_{jk}^{*}(-\theta) \qquad \text{and} \qquad g_{kj}(\theta) = g_{\bar{k}j}(i\pi - \theta) \qquad (3.12)$$

hold. The unitarity requirement is satisfied once we fulfill the functional relation

$$g_{ij}(\theta) g_{ji}(-\theta) = \left[1 - \kappa^2 m_i m_j \left(\frac{\left(\rho_{ij}^{+}\right)^2 + \sinh^2\frac{\theta}{2}}{\sinh^2\theta} \right) \right]^{-1} =: \chi_{ij}(\theta)\,. \qquad (3.13)$$

In order to solve the set of equations (3.12)–(3.13) we assume now first parity invariance for g and self-conjugacy for the particles involved

$$g_{kj}(\theta) = g_{jk}(\theta) \qquad \text{and} \qquad g_{jk}(\theta) = g_{j\bar{k}}(\theta)\,. \tag{3.14}$$

Following a standard procedure to solve functional equations of the above type we make the general ansatz

$$g_{ij}(\theta) = \lambda \prod_{l=1}^{\infty} \frac{\rho_{ij}[\theta + 2\pi i l]\rho_{ij}[-\theta + 2\pi i(l+1/2)]}{\rho_{ij}[\theta + 2\pi i(l+1/2)]\rho_{ij}[-\theta + 2\pi i(l+1)]}\,. \tag{3.15}$$

At this stage $\lambda \in \mathbb{C}$ is some arbitrary constant and the ρ_{ij} are some functions which still need to be determined. The ansatz (3.15) solves the crossing relation (3.12) by construction when also (3.14) holds. Substituting (3.15) into (3.13) we then find that

$$\chi_{ij}(\theta) = \lambda^2 \rho_{ij}\,(\theta + 2\pi i)\,\rho_{ij}\,(2\pi i - \theta) \tag{3.16}$$

has to be satisfied. Hence, we have reduced the problem of simultaneously solving (3.12) and (3.13) to a much simpler problem of just factorizing the function χ. Unfortunately, (3.13) can not yet be compared directly with (3.16), but it was noted in [8] that when introducing two auxiliary equations which parameterize the masses and κ in terms of the new quantities η_{ij}, $\hat{\eta}_{ij}$

$$\frac{\kappa^2}{2} m_i m_j = \cos\eta_{ij} + \cos\hat{\eta}_{ij} \qquad \text{and} \qquad -\frac{\kappa^2}{4}\left(m_i^2 + m_j^2\right) = 1 + \cos\eta_{ij}\cos\hat{\eta}_{ij}\,, \tag{3.17}$$

one can bring χ into a more suitable form

$$\chi_{ij}(\theta) = \frac{\sinh^2\frac{\theta}{2}\cosh^2\frac{\theta}{2}}{\sinh\frac{1}{2}(\theta + i\eta_{ij})\sinh\frac{1}{2}(\theta - i\eta_{ij})\sinh\frac{1}{2}(\theta + i\hat{\eta}_{ij})\sinh\frac{1}{2}(\theta - i\hat{\eta}_{ij})}\,. \tag{3.18}$$

Comparing now (3.16) and (3.18) there are obviously various solutions. Starting by producing the factors $\pi^2/\sinh\frac{1}{2}(\theta + i\eta)\sinh\frac{1}{2}(\theta - i\eta)$ for $\eta = \eta_{ij}, \hat{\eta}_{ij}$, we have the possibilities

$$\rho_{ij}^{(1/2)}(\theta + 2\pi i, \eta) = \Gamma\left(\frac{i\theta \mp \eta}{2\pi}\right)\Gamma\left(1 + \frac{i\theta \pm \eta}{2\pi}\right), \tag{3.19}$$

$$\rho_{ij}^{(3/4)}(\theta + 2\pi i, \eta) = \Gamma\left(\frac{-i\theta \mp \eta}{2\pi}\right)\Gamma\left(1 - \frac{i\theta \mp \eta}{2\pi}\right), \tag{3.20}$$

$$\rho_{ij}^{(5/6)}(\theta + 2\pi i, \eta) = \pm\pi/\sinh\frac{1}{2}(\theta \pm i\eta)\,. \tag{3.21}$$

We can now substitute these solutions back into (3.15) in order to assemble $g_{ij}(\theta, \eta)$. When restricting w.l.g. the parameters $0 < \eta_{ij}, \hat{\eta}_{ij} < \pi$, we observe that all functions $g_{ij}(\theta, \eta)$ have poles inside the physical sheet, that is $0 < \operatorname{Im}\theta \leq \pi$, except the one constructed from $\rho_{ij}^{(4)}(\theta, \eta)$. Thus only for this solution the boson-fermion mixing factor $\check{S}$ does not introduce new bound states (see next subsection for more details on fusing), such that the fusing structure is entire contained in the bosonic factor $\hat{S}$. Selecting out this particular solution we can write

$$g_{ij}(\theta) = \frac{1}{2i}\frac{\sinh\theta}{\sinh\frac{1}{2}(\theta + i\eta_{ij})\sinh\frac{1}{2}(\theta + i\hat{\eta}_{ij})} g(\theta, \eta_{ij}) g(\theta, \hat{\eta}_{ij}) \tag{3.22}$$

where we defined the function

$$g(\theta,\eta) = i \prod_{k=1}^{\infty} \frac{\Gamma\left(k - \frac{i\theta+\eta}{2\pi}\right)\Gamma\left(k + \frac{1}{2} + \frac{i\theta-\eta}{2\pi}\right)\Gamma\left(k - \frac{i\theta-\eta}{2\pi}\right)\Gamma\left(k - \frac{1}{2} + \frac{i\theta+\eta}{2\pi}\right)}{\Gamma\left(k + \frac{i\theta-\eta}{2\pi}\right)\Gamma\left(k + \frac{1}{2} - \frac{i\theta+\eta}{2\pi}\right)\Gamma\left(k + \frac{i\theta+\eta}{2\pi}\right)\Gamma\left(k - \frac{1}{2} - \frac{i\theta-\eta}{2\pi}\right)}$$

$$= \exp\left[\int_0^\infty \frac{dt}{t}\left[\frac{\sinh t\left(\frac{1}{2} - \frac{\eta}{\pi}\right)}{2\sinh\frac{t}{2}\cosh^2 t/2} - 1\right]\sinh\frac{t\theta}{2\pi i}\right]. \tag{3.23}$$

Clearly it would be very interesting to investigate also the solutions resulting from the functions (3.19)–(3.21) other than $\rho_{ij}^{(4)}$. Further solutions can be expected when one relaxes the assumptions (3.14).

3.4 Constraints from the boundstate bootstrap equations

The last remaining constraint arises when we consider the consequences of the factorization of the S-matrix in conjunction with the possibility of a fusing process, say $\mu_i + \nu_j \to \bar{\kappa}_k$, for $\mu, \nu, \kappa = b, f$ and $1 \leq i, j, k \leq \ell$. For this to happen the scattering matrix must posses a simple order pole in the physical sheet at some fusing angle $i\eta_{\mu_i\nu_j}^{\bar{\kappa}_k}$ with $\eta_{\mu_i\nu_j}^{\bar{\kappa}_k} \in \mathbb{R}^+$. The residue of S at this angle is related to the three-point couplings $\Gamma_{\mu_i\nu_j}^{\bar{\kappa}_k}$ via

$$i \operatorname*{Res}_{\theta \to i\eta_{\mu_i\nu_j}^{\bar{\kappa}_k}} S_{\mu_i\nu_j}^{\rho_l\tau_m}(\theta) = \sum_{\bar{\kappa}_k} \left(\Gamma_{\rho_l\tau_m}^{\bar{\kappa}_k}\right)^* \Gamma_{\mu_i\nu_j}^{\bar{\kappa}_k}. \tag{3.24}$$

Then the following boundstate bootstrap equation [43, 52, 53, 46]

$$\sum_{\delta_d,\gamma_g,\rho_n} \Gamma_{\delta_d\rho_n}^{\beta_b} S_{\mu_i\gamma_g}^{\alpha_a\delta_d}(\theta + i\bar{\eta}_{\bar{\kappa}_k\mu_i}^{\bar{\nu}_j}) S_{\nu_j\lambda_l}^{\gamma_g\rho_n}(\theta - i\bar{\eta}_{\nu_j\kappa_k}^{\bar{\mu}_i}) = \sum_{\bar{\kappa}_k} S_{\bar{\kappa}_k\lambda_l}^{\alpha_a\beta_b}(\theta)\Gamma_{\mu_i\nu_j}^{\bar{\kappa}_k} \tag{3.25}$$

has to be satisfied. Here the $\bar{\eta}$ is related to the fusing angle η as $\bar{\eta} = \pi - \eta$. Taking the factorization ansatz (3.1) for S into account and assuming further that the bosonic part of the scattering matrix is diagonal $\hat{S}_{ij}^{kl}(\theta) = \delta_i^l\delta_j^k\hat{S}_{ij}(\theta)$, the relation (3.25) simplifies to

$$\sum_{\delta,\gamma,\rho} \Gamma_{\delta_i\rho_j}^{\beta_b} S_{\mu_i\gamma_l}^{\alpha_l\delta_i}(\theta + i\bar{\eta}_{\bar{\kappa}_k\mu_i}^{\bar{\nu}_j}) S_{\nu_j\lambda_l}^{\gamma_l\rho_j}(\theta - i\bar{\eta}_{\nu_j\kappa_k}^{\bar{\mu}_i}) = \sum_{\bar{\kappa}} S_{\bar{\kappa}_b\lambda_l}^{\alpha_l\beta_b}(\theta)\Gamma_{\mu_i\nu_j}^{\bar{\kappa}_b}. \tag{3.26}$$

It is not difficult to convince oneself that (3.26) results from the formal equation

$$Z_{\mu_i}\left(\theta + i\bar{\eta}_{\mu_k\mu_i}^{\bar{\mu}_j}\right) Z_{\nu_j}\left(\theta - i\bar{\eta}_{\mu_j\mu_k}^{\bar{\mu}_i}\right) = \sum_{\bar{\kappa}_k} \Gamma_{\mu_i\nu_j}^{\bar{\kappa}_k} Z_{\bar{\kappa}_k}(\theta). \tag{3.27}$$

together with the assumption that the Zs obey a Zamolodchikov algebra [54], i.e. when exchanging (braiding) them they will pick up an S-matrix as a structure constant. Acting on (3.27) with $\hat{\mathcal{Q}}$ one notices first of all that only the following fusing processes are allowed to occur

$$b_i + b_j \to b_k, \qquad f_i + f_j \to b_k, \qquad b_i + f_j \to f_k, \qquad f_i + b_j \to f_k. \tag{3.28}$$

Furthermore when acting with $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ on (3.27) one finds a powerful constraint for the three point couplings

$$\left(\frac{\Gamma_{b_i b_j}^{b_k}}{\Gamma_{f_i f_j}^{b_k}}\right)^2 = \frac{m_k + m_i + m_j}{m_i + m_j - m_k} . \tag{3.29}$$

Computing then the residues of S by means of (3.24) for the processes $b_i + b_j \to b_k$ and $f_i + f_j \to b_k$ yields for the constant κ in (3.10) the relation

$$\kappa = \frac{\sin \eta_{ij}^k}{\sqrt{m_i m_j} \rho_{ij}^+} \left[\frac{\left(\Gamma_{b_i b_j}^{b_k}\right)^2 + \left(\Gamma_{f_i f_j}^{b_k}\right)^2}{\left(\Gamma_{b_i b_j}^{b_k}\right)^2 - \left(\Gamma_{f_i f_j}^{b_k}\right)^2} \right] = \frac{2 \sin \eta_{ij}^k}{m_k} . \tag{3.30}$$

Notice that this is quite a severe constraint as the right hand side of (3.30) has to hold universally for all possible values of i, j, k.

4. Implementing unstable particles

As a consequence of the factorizing ansatz (3.1) for S and the choice for $\check{S}$ which does not possess poles inside the physical sheet, the pole structure responsible for fusing processes is entirely confined to the bosonic factor $\hat{S}$. One may therefore search the large reservoir of diagonal S-matrices to find suitable solutions. In the original paper Schoutens [8] noticed that one may satisfy (3.30) with $\hat{S}$ equal to the scattering matrix of minimal $A_{2\ell}^{(2)}$-affine Toda field theory [55]

$$S_{ab}(\theta) = \left(\frac{a+b}{2n+1}\right)_\theta \left(\frac{|a-b|}{2n+1}\right)_\theta \prod_{k=1}^{\min(a,b)-1} \left(\frac{a+b-2k}{2n+1}\right)_\theta^2 \tag{4.1}$$

for $1 \leq a, b \leq \ell$ and with $(x)_\theta := \tanh \frac{1}{2}(\theta + i\pi x) / \tanh \frac{1}{2}(\theta - i\pi x)$. Thereafter, Hollowood and Mavrikis [12] showed that when taking the bosonic factor $\hat{S}$ to be the minimal $A_\ell^{(1)}$, $D_\ell^{(1)}$ or $(C_\ell^{(1)}|D_{\ell+2}^{(2)})$-affine Toda S-matrix, the ansatz (3.1) also satisfies (3.30) together with the bootstrap equations, thus leading to consistent supersymmetric S-matrices.

Based on these results it is straightforward to extend the ansatz and also include unstable particles into the spectrum of these theories. We may take the bosonic factor of $\hat{S}$ to belong to the large class of models which can be referred to conveniently as $\mathbf{g}|\tilde{\mathbf{g}}$-theories. In these models each particle carries two quantum numbers (a, i), one associated to the algebra $\mathbf{g}$ with $1 \leq a \leq \ell = \operatorname{rank} \mathbf{g}$ and the other related to the algebra $\tilde{\mathbf{g}}$ with $1 \leq i \leq \tilde{\ell} = \operatorname{rank} \tilde{\mathbf{g}}$. We then argue that scattering matrices of the general form

$$S_{\alpha_{(a,i)}\beta_{(b,j)}}^{\gamma_{(b,j)}\delta_{(a,i)}}(\theta, \sigma_{ij}) = \hat{S}_{ab}^{ij}(\theta, \sigma_{ij}) \check{S}_{\alpha_a \beta_b}^{\gamma_b \delta_a}(\theta) \tag{4.2}$$

will also satisfy all the above mentioned constraints and constitute therefore consistent scattering matrices which are by construction invariant under supersymmetry and allow unstable particles in their spectrum.

JHEP01(2005)030

The general formula for $\mathbf{g}|\tilde{\mathbf{g}}$-scattering matrices in form of an integral representation [21] is

$$\hat{S}_{ab}^{ij}(\theta,\sigma_{ij}) = \eta_{ab}^{ij} \exp \int_{-\infty}^{\infty} \frac{dt}{t} \hat{\Phi}(t,h) e^{-it(\theta+\sigma_{ij})}, \qquad \eta_{ab}^{ij} = \exp\left(i\pi\varepsilon_{ij}[K^{-1}]_{\bar{a}b}\right) \quad (4.3)$$

$$\hat{\Phi}_{ab}^{ij}(t) = \delta_{ab}\delta_{ij} - \left(2\cosh\frac{\pi t}{h} - \tilde{I}\right)_{ij} \left(2\cosh\frac{\pi t}{h} - I\right)_{ab}^{-1}. \quad (4.4)$$

Here we denote by I ($\tilde{I}$) and K ($\tilde{K}$) the incidence and Cartan matrices for the simply laced $\mathbf{g}$ ($\tilde{\mathbf{g}}$)-Lie algebra, respectively. The Coxeter number of $\mathbf{g}$ ($\tilde{\mathbf{g}}$) is h ($\tilde{h}$) and $\varepsilon_{ij} = -\varepsilon_{ji}$ is the Levi-Civita pseudo-tensor. The special cases $\mathbf{A}_\ell|\tilde{\mathbf{g}}$ and $\mathbf{g}|\mathbf{A}_1$ correspond to the $\tilde{\mathbf{g}}_{\ell+1}$-homogeneous sine-Gordon models [20] and $\mathbf{g}$ -minimal affine Toda field theories (see e.g. [56] for a complete list), respectively. In the ultraviolet limit these models reduce to conformal field theories, which were discussed in [57], possessing Virasoro central charges $c^{\mathbf{g}|\tilde{\mathbf{g}}} = \ell\tilde{\ell}\tilde{h}/(h+\tilde{h})$. Besides simply laced Lie algebras, we will here also allow $\mathbf{g}$ and $\tilde{\mathbf{g}}$ to be the twisted algebra $A_{2\ell}^{(2)}$. These cases have not been considered previously. We have verified here for various examples that the previous formula for the Viasoro central charge also applies when including $A_{2\ell}^{(2)}$ with rank ℓ and $h = 2\ell+1$ (see below).

The novel feature in S-matrices of the type (4.3) is the occurrence of the resonance parameters $\sigma_{ij} = -\sigma_{ji}$. Besides the first order poles in the physical sheet which can be interpreted as bound states of stable particles, there are also simple order poles in the second Riemann sheet at $\theta_{ab} = -i\eta_{ab}^{\bar{c}} + \sigma_{ab}^{\bar{c}}$ with $\eta_{ab}^{\bar{c}}$, $\sigma_{ab}^{\bar{c}} \in \mathbb{R}^+$. Poles of this type admit an interpretation as unstable particles of type $\bar{c}$ with finite lifetime $\tau_{\bar{c}}$. The relations between the masses of the stable particles m_a, m_b, the mass of the unstable particle $m_{\bar{c}}$ and the fusing angles $\eta_{ab}^{\bar{c}}$, $\sigma_{ab}^{\bar{c}}$ are the Breit-Wigner equations [58]

$$m_{\bar{c}}^2 - \frac{1}{(4\tau_{\bar{c}}^2)} = m_a^2 + m_b^2 + 2m_a m_b \cosh\sigma_{ab}^{\bar{c}} \cos\eta_{ab}^{\bar{c}}, \quad (4.5)$$

$$\frac{m_{\bar{c}}}{\tau_{\bar{c}}} = 2m_a m_b \sinh\sigma_{ab}^{\bar{c}} \sin\eta_{ab}^{\bar{c}}. \quad (4.6)$$

The ansatz (4.2) for the choices $\mathbf{g} = \{A_{2\ell}^{(2)}, A_\ell^{(1)}, D_\ell^{(1)}, (C_\ell^{(1)}|D_{\ell+2}^{(2)})\}$ and $\tilde{\mathbf{g}}$ being any simple Lie algebra satisfies all consistency conditions, in particular the bootstrap equation. When including non-simply laced Lie algebras, but $A_{2\ell}^{(2)}$, we also need to take some modifications into account [22], from which we refrain here in order to keep the notation simple.

The procedure to obtain the supersymmetic version for models which are constructed by an analytic continuation of the effective coupling constant [32]–[42] is staightforward, as one simply has to carry out this continuation for the supersymmetric theory with real coupling.

5. The ultraviolet limit, a TBA analysis

Let us now carry out the ultraviolet limit by means of a thermodynamic Bethe ansatz (TBA) analysis [59] for the above mentioned scattering matrices by following the work of

JHEP01(2005)030

Ahn [60], Moriconi and Schoutens [61]. In general the TBA is technically very complicated when involving non-diagonal S-matrices. Fortunately, for the case at hand matters simplify drastically due to the fact that $\hat{S}$ satisfies the so-called free fermion condition [60, 61] (this is a rather misleading terminology as we are evidently not dealing with free fermions). Here we are only interested in the extreme ultraviolet limit for which there exists a standard analysis [59], which can be adapted to the supersymmetric case [61]. We restrict the following analysis to the ansatz (4.2) for a $\mathbf{g}|\tilde{\mathbf{g}}$-supersymmetric S-matrix with unstable particles where we take $\mathbf{g} = A_{2\ell}^{(2)}$. Following [60, 61] it is straightforward to derive the constant TBA equation for the S-matrix (4.2) with the quoted choice of the algebras and all resonance parameters σ_{ij} set to zero

$$x_a^i = (1+x_0^i)^{\check{M}_a} \prod_{j=1}^{\tilde{\ell}} \prod_{b=1}^{\ell} (1+x_b^j)^{N_{ab}^{ij}} \quad x_0^i = \prod_{b=1}^{\ell} (1+x_b^i)^{\check{M}_b} \qquad \text{for } 1 \leq i \leq \tilde{\ell},\ 1 \leq a \leq \ell\,. \tag{5.1}$$

The matrices in (5.1) are computed from

$$\hat{N}_{ab}^{ij} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta \hat{\Phi}_{ab}^{ij}(\theta) = \delta_{ab}\delta_{ij} - \min(a,b)\tilde{K}_{ij}, \tag{5.2}$$

$$\check{N}_{ab} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta \check{\Phi}_{ab}(\theta) = \frac{1}{2}, \tag{5.3}$$

$$\check{M}_a = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta \varphi_a(\theta) = 1\,, \tag{5.4}$$

$$N_{ab}^{ij} = \hat{N}_{ab}^{ij} + \check{N}_{ab} - \frac{1}{2}\check{M}_a\check{M}_b = \delta_{ab}\delta_{ij} - \min(a,b)\tilde{K}_{ij}\,, \tag{5.5}$$

with kernels

$$\check{\Phi}_{ab}(\theta) = \operatorname{Im} \frac{\partial}{\partial\theta} \ln \left[\frac{g_{ab}(\theta)}{\sinh\theta} \right], \tag{5.6}$$

$$\varphi_a(\theta) = 2\operatorname{Im} \frac{\partial}{\partial\theta} \ln \left[\sinh\frac{1}{2}\left(\theta - \frac{i\pi a}{h}\right) \cosh\frac{1}{2}\left(\theta + \frac{i\pi a}{h}\right) \right]. \tag{5.7}$$

The expression (5.2) results directly from (4.3), (4.4) when noting that the entries of inverse Cartan matrix of $A_{2\ell}^{(2)}$ are $\min(a,b)$ with $1 \leq a,b \leq \ell$. From the solution for the function $g(\theta)$ in the form (3.23) we compute the constant (5.3). Note that the final answer does not depend on the quantities η_{ij}, $\hat{\eta}_{ij}$ which were introduced in (3.17) to parameterize the masses and the constant κ. The constants $\check{M}_a$ are obtained by direct computation and hold for all $1 \leq a \leq \ell$. Assembling then all quantities in (5.5) one observes that all contributions resulting from the supersymmetric factor of the S-matrix have cancelled out, such that $N_{ab}^{ij} = \hat{N}_{ab}^{ij}$. Hence (5.1) resembles very closely the conventional, that is non-supersymmetric, constant TBA equations with the modification of the factor involving an additional particle, named 0, which results from the diagonalization procedure.

Having solved (5.1) one can compute the effective Virasoro central charge as

$$c_{\text{eff}} = \frac{6}{\pi^2} \sum_{k=0}^{\ell} \sum_{j=1}^{\tilde{\ell}} \left[\mathcal{L}\left(\frac{x_k^j}{1+x_k^j}\right) - \mathcal{L}\left(\frac{y_k^j}{1+y_k^j}\right) \right], \tag{5.8}$$

with $\mathcal{L}(x)$ denoting Rogers dilogarithm $\mathcal{L}(x) = \sum_{n=1}^{\infty} x^n/n^2 + \ln x \ln(1-x)/2$, $x_a^i = \exp(-\varepsilon_a^i(0))$, $y_a^i = \lim_{\theta\to\infty}\exp(-\varepsilon_a^i(\theta))$ and $\varepsilon_a^i(\theta)$ being the rapidity dependent pseudo-energies.

5.1 The $\mathbf{A}_{2\ell}^{(2)}|\tilde{\mathbf{g}}$-theories

Even though our main goal is to discuss the supersymmetric scenario, we shall comment first on the solutions of (5.1) and the subsequent computation of c_{eff} in the absence of supersymmetry involving the twisted algebra $\mathbf{A}_{2\ell}^{(2)}$. The reason for this is that this case will be needed below and has hitherto not been dealt with in the literature. The supersymmetry is formally broken in (5.1) when taking the limit $\check{M}_a \to 0$ for all $a \in \{1,\ldots,\ell\}$. Selecting the algebra which encodes the unstable particles to be $\tilde{\mathbf{g}} = A_{\tilde{\ell}}^{(1)}$, we found the following analytic solutions for (5.1)

$$x_a^j = \frac{\sin j\pi/\tau \sin(\tilde{h}-j)\pi/\tau}{\sin a\pi/\tau \sin(\tilde{h}+a)\pi/\tau} \qquad \text{for } 1 \le a \le \ell,\ 1 \le j \le \tilde{\ell}, \tag{5.9}$$

where h ($\tilde{h}$) is the Coxeter number of $A_{2\ell}^{(2)}$ ($A_{\tilde{\ell}}^{(1)}$), namely $h = 2\ell+1$ ($h = 2\tilde{\ell}+1$) and $\tau = h + \tilde{h}$. In fact, the solutions (5.9) for the constant TBA-equations (5.1) hold for all four $\mathbf{g}|\tilde{\mathbf{g}}$-theories with $\mathbf{g},\tilde{\mathbf{g}} \in \{A_n^{(1)}, A_{2n}^{(2)}\}$. Computing the effective central charge by means of (5.8) yields the usual value [21] of the $\mathbf{g}|\tilde{\mathbf{g}}$-theories

$$c_{\text{eff}} = \frac{\ell\tilde{\ell}\tilde{h}}{h+\tilde{h}}\,. \tag{5.10}$$

We have solved (5.1) for other $\mathbf{A}_{2\ell}^{(2)}|\tilde{\mathbf{g}}$-theories involving various simply laced algebras $\tilde{\mathbf{g}}$ and obtained (5.10) in all cases. So far we have not found simple closed expressions for the x_a^j as in (5.9) for these cases and will not present here more case-by-case results.

5.2 The $N=1$ supersymmetric $\mathbf{A}_{2\ell}^{(2)}|\tilde{\mathbf{g}}$-theories

We shall now turn to the full supersymmetric version of the constant TBA-equations (5.1) describing the $\mathbf{A}_{2\ell}^{(2)}|\tilde{\mathbf{g}}$-theories. In general, we may write (5.1) as

$$x_a^i = (1+x_0^i)\prod_{j=1}^{\tilde{\ell}}(1+x_1^j)^{N_{a1}^{ij}}\prod_{j=1}^{\tilde{\ell}}\prod_{b=2}^{\ell}(1+x_b^j)^{N_{ab}^{ij}} \qquad \text{for } 1 \le i \le \tilde{\ell},\ 1 \le a \le \ell\,. \tag{5.11}$$

Excluding $a=1$ and noting the simple fact that $\min(a,b) = \min(a-1,b-1)+1$ we may re-write (5.11) for the theories at hand as

$$x_a^i = (1+x_0^i)\prod_{j=1}^{\tilde{\ell}}(1+x_1^j)^{-K_{ij}}\prod_{b=2}^{\ell}(1+x_b^j)^{-K_{ij}}(1+x_b^j)^{N_{(a-1)(b-1)}^{ij}} \qquad \text{for } 2 \le a \le \ell\,. \tag{5.12}$$

Taking the limit $x_1^i \to \infty$ of this equation leads to

$$x_a^i = \lim_{x_1^i\to\infty}\left[x_1^i\prod_{j=1}^{\tilde{\ell}}(x_1^j)^{-K_{ij}}\prod_{b=2}^{\ell}(1+x_b^i)(1+x_b^j)^{-K_{ij}}(1+x_b^j)^{N_{(a-1)(b-1)}^{ij}}\right]. \tag{5.13}$$

For $\tilde{\mathbf{g}} = A_2^{(1)}$ we may assume that $x_a^1 = x_a^2$, such that (5.13) simplifies to

$$x_a^i = \prod_{j=1}^{\tilde{\ell}} \prod_{b=2}^{\ell} (1 + x_b^j)^{N^{ij}_{(a-1)(b-1)}} \qquad \text{for } 1 \leq i \leq \tilde{\ell},\ 2 \leq a \leq \ell\,. \tag{5.14}$$

which is precisely the system for an $\mathbf{A}^{(2)}_{2(\ell-1)}|A_2^{(1)}$-theory when renaming the particles $a = 2$ to $a = 1$, $a = 3$ to $a = 2$, $\ldots$, $a = \ell$ to $a = \ell - 1$. The solutions for the constant TBA equations (5.1) are therefore in this case $x_0^1 = x_0^2 = x_1^1 = x_2^1 \to \infty$ and $x_k^1 = x_k^2$ for $1 \leq k \leq \ell - 1$ given by (5.9). Taking further $y_0^1 = y_0^2 = 1$, $y_k^1 = y_k^2 = 0$ for $1 \leq k \leq \ell - 1$ the effective central charge is then computed by means of (5.8)

$$c_{\text{eff}} = \frac{6}{\pi^2}\left[\sum_{k=1}^{\ell-1}\sum_{j=1}^{2} \mathcal{L}\left(\frac{x_k^j}{1+x_k^j}\right) + 4\mathcal{L}(1) - 2\mathcal{L}\left(\frac{1}{2}\right)\right] = \frac{3(\ell-1)}{\ell+1} + 3 = \frac{6\ell}{\ell+1}\,. \tag{5.15}$$

We may also investigate the behaviour of these theories for large resonance parameters. As the bosonic part of the theory decouples in this case into two separate non-interacting theories [23, 31, 25], the two $N = 1$ supersymmetric theories will behave analogously due to the factorization ansatz for S. Accordingly we have

$$\lim_{\sigma_{12}\to\infty} \mathbf{A}^{(2)}_{2\ell}|A_2^{(1)} \to \mathbf{A}^{(2)}_{2\ell}|A_1^{(1)} \otimes \mathbf{A}^{(2)}_{2\ell}|A_1^{(1)}\,. \tag{5.16}$$

The resulting $\mathbf{A}^{(2)}_{2\ell}|A_1^{(1)}$-theories are the $N = 1$ supersymmetric theories discussed in [8, 61]. The effective Virasoro central charge is then simply obtained as the sum of the known effective central charges of the supersymmetric minimal $\mathcal{SM}(2, 4\ell + 4)$ conformal field theories

$$c_{\text{eff}} = \frac{3\ell}{2\ell+2} + \frac{3\ell}{2\ell+2} = \frac{3\ell}{\ell+1}\,. \tag{5.17}$$

Thus we observe that the effective central charge for the theory with vanishing resonance parameter is twice the one with large resonance parameter.

Clearly it is interesting to carry out the TBA analysis for other algebras $\tilde{\mathbf{g}}$. Here it suffices to have demonstrated that the proposed scattering matrices of the type (4.2) have a meaningful ultraviolet behaviour, which in the presented cases can even be obtained analytically.

6. Conclusion

We have shown that our S-matrix proposal (4.2) consistently combines $N = 1$ supersymmetry with the requirement to have unstable particle in the spectrum of the theory. The S-matrix satisfies all the constraints imposed by the bootstrap program and possesses a sensible ultraviolet limit.

There are various open issues which would be interesting to address in future. The proposal (4.2) constitutes the first concrete example for a non-diagonal scattering matrix corresponding to a theory which contains unstable particles. It would be interesting to construct further non-diagonal scattering matrices of this type for which the supersymmetry is broken or possibly enlarged to greater values of N.

Clearly it would be interesting to complete the detailed analysis involving also other algebras on the bosonic side. More challenging is to modify the boson-fermion mixing part. So far the entire fusing structure of the model was confined to the bosonic factor. However, we also provided solutions for the function $g(\theta)$ which has simple poles in the physical sheet, which can be interpreted as stable bound states possibly leading to consistent solutions for the bootstrap equations. Concerning the implementation of unstable particles, it should also be possible to extend the ansatz (4.2) to the form

$$S_{\alpha_{(a,i)}\beta_{(b,j)}}^{\gamma_{(b,j)}\delta_{(a,i)}}(\theta,\sigma_{ij}) = \hat{S}_{ab}^{ij}(\theta,\sigma_{ij})\check{S}_{\alpha_{(a,i)}\beta_{(b,j)}}^{\gamma_{(b,j)}\delta_{(a,i)}}(\theta,\sigma_{ij})\,. \tag{6.1}$$

This would means that the unstable particles are be no longer of a purely bosonic nature. More general alterations such as taking the bosonic factor to be non-diagonal or relaxing the factorzation ansatz into a purely bosonic and boson-fermion mixing factor altogether have not even been considered in the absence of unstable particles.

Acknowledgments

I am grateful to M. Moriconi for useful discussions and to the members of the Institut für theoretische Physik at the University of Hannover for kind hospitality.

References

[1] Y.A. Golfand and E.P. Likhtman, *Extension of the algebra of Poincare group generators and violation of P invariance*, *JETP Lett.* **13** (1971) 323.

[2] P. Ramond, *Dual theory for free fermions*, *Phys. Rev.* **D 3** (1971) 2415.

[3] A. Neveu and J.H. Schwarz, *Factorizable dual model of pions*, *Nucl. Phys.* **B 31** (1971) 86.

[4] F. Gliozzi, J. Scherk and D.I. Olive, *Supersymmetry, supergravity theories and the dual spinor model*, *Nucl. Phys.* **B 122** (1977) 253.

[5] J. Wess and B. Zumino, *Supergauge transformations in four-dimensions*, *Nucl. Phys.* **B 70** (1974) 39.

[6] S.R. Coleman and J. Mandula, *All possible symmetries of the S matrix*, *Phys. Rev.* **159** (1967) 1251.

[7] R. Shankar and E. Witten, *The s matrix of the supersymmetric nonlinear sigma model*, *Phys. Rev.* **D 17** (1978) 2134.

[8] K. Schoutens, *Supersymmetry and factorizable scattering*, *Nucl. Phys.* **B 344** (1990) 665.

[9] P. Fendley, *A second supersymmetric s matrix for the perturbed tricritical ising model*, *Phys. Lett.* **B 250** (1990) 96.

[10] C. Ahn, *Complete s matrices of supersymmetric sine-Gordon theory and perturbed superconformal minimal model*, *Nucl. Phys.* **B 354** (1991) 57.

[11] J.M. Evans and T.J. Hollowood, *Exact scattering in the* SU(N) *supersymmetric principal chiral model*, *Nucl. Phys.* **B 493** (1997) 517 [hep-th/9603190].

[12] T.J. Hollowood and E. Mavrikis, *The $N = 1$ supersymmetric bootstrap and Lie algebras*, *Nucl. Phys.* **B 484** (1997) 631 [hep-th/9606116].

[13] G.M. Gandenberger, *Trigonometric s-matrices, affine Toda solitons and supersymmetry*, *Int. J. Mod. Phys.* **A 13** (1998) 4553 [hep-th/9703158].

[14] G. Mussardo, *Exact matrix elements in supersymmetric theories*, *Nucl. Phys.* **B 532** (1998) 529 [hep-th/9806184].

[15] R. Köberle and V. Kurak, *The S matrix of a factorizable supersymmetric Z_N model*, *Phys. Lett.* **B 191** (1987) 295.

[16] P. Fendley, S.D. Mathur, C. Vafa and N.P. Warner, *Integrable deformations and scattering matrices for the $N = 2$ supersymmetric discrete series*, *Phys. Lett.* **B 243** (1990) 257.

[17] K. Kobayashi and T. Uematsu, *S matrix of $N = 2$ supersymmetric sine-Gordon theory*, *Phys. Lett.* **B 275** (1992) 361 [hep-th/9110040].

[18] P. Fendley and K.A. Intriligator, *Scattering and thermodynamics of fractionally charged supersymmetric solitons*, *Nucl. Phys.* **B 372** (1992) 533 [hep-th/9111014].

[19] A.B. Zamolodchikov, *Z(4) symmetric factorized S matrix in two space-time dimensions*, *Commun. Math. Phys.* **69** (1979) 165.

[20] J.L. Miramontes and C.R. Fernandez-Pousa, *Integrable quantum field theories with unstable particles*, *Phys. Lett.* **B 472** (2000) 392 [hep-th/9910218].

[21] A. Fring and C. Korff, *Colour valued scattering matrices*, *Phys. Lett.* **B 477** (2000) 380 [hep-th/0001128].

[22] C. Korff, *Colour valued scattering matrices from non simply-laced Lie algebras*, *Phys. Lett.* **B 501** (2001) 289 [hep-th/0010287].

[23] O.A. Castro-Alvaredo, A. Fring, C. Korff and J.L. Miramontes, *Thermodynamic Bethe ansatz of the homogeneous sine-Gordon models*, *Nucl. Phys.* **B 575** (2000) 535 [hep-th/9912196].

[24] P. Dorey and J.L. Miramontes, *Aspects of the homogeneous sine-Gordon models*, in *Workshop on integrable theories, solitons and duality*, Sao Paulo, Brazil 2002.

[25] O.A. Castro-Alvaredo, J. Dreißig and A. Fring, *Integrable scattering theories with unstable particles*, *Eur. Phys. J.* **C 35** (2004) 393 [hep-th/0211168].

[26] O. Castro-Alvaredo and A. Fring, *On vacuum energies and renormalizability in integrable quantum field theories*, *Nucl. Phys.* **B 687** (2004) 303 [hep-th/0401075].

[27] P. Dorey and J.L. Miramontes, *Mass scales and crossover phenomena in the homogeneous sine-Gordon models*, *Nucl. Phys.* **B 697** (2004) 405 [hep-th/0405275].

[28] O.A. Castro-Alvaredo, A. Fring and C. Korff, *Form factors of the homogeneous sine-Gordon models*, *Phys. Lett.* **B 484** (2000) 167 [hep-th/0004089].

[29] O.A. Castro-Alvaredo and A. Fring, *Renormalization group flow with unstable particles*, *Phys. Rev.* **D 63** (2001) 021701 [hep-th/0008208].

[30] O.A. Castro-Alvaredo and A. Fring, *Identifying the operator content, the homogeneous sine-Gordon models*, *Nucl. Phys.* **B 604** (2001) 367 [hep-th/0008044].

[31] O.A. Castro-Alvaredo and A. Fring, *Decoupling the $\mathrm{SU}(N)_2$ homogeneous sine-Gordon model*, *Phys. Rev.* **D 64** (2001) 085007 [hep-th/0010262].

[32] A.B. Zamolodchikov, *Resonance factorized scattering and roaming trajectories*, Preprint ENS-LPS-335, 1991.

[33] M.J. Martins, *Renormalization group trajectories from resonance factorized S matrices*, *Phys. Rev. Lett.* **69** (1992) 2461 [hep-th/9205024].

[34] M.J. Martins, *Exact resonance A-D-E s matrices and their renormalization group trajectories*, *Nucl. Phys.* **B 394** (1993) 339 [hep-th/9208011].

[35] P.E. Dorey and F. Ravanini, *Staircase models from affine Toda field theory*, *Int. J. Mod. Phys.* **A 8** (1993) 873 [hep-th/9206052].

[36] P. Dorey and F. Ravanini, *Generalizing the staircase models*, *Nucl. Phys.* **B 406** (1993) 708 [hep-th/9211115].

[37] C. Ahn, G. Delfino and G. Mussardo, *Mapping between the sinh-Gordon and Ising models*, *Phys. Lett.* **B 317** (1993) 573 [hep-th/9306103].

[38] G. Mussardo and S. Penati, *A quantum field theory with infinite resonance states*, *Nucl. Phys.* **B 567** (2000) 454 [hep-th/9907039].

[39] O.A. Castro-Alvaredo and A. Fring, *Constructing infinite particle spectra*, *Phys. Rev.* **D 64** (2001) 085005 [hep-th/0103252].

[40] O.A. Castro-Alvaredo and A. Fring, *Breathers in the elliptic sine-Gordon model*, *J. Phys.* **A36** (2003) 10233 [hep-th/0303245].

[41] A. Leclair, J.M. Roman and G. Sierra, *Russian doll renormalization group, Kosterlitz-Thouless flows and the cyclic sine-Gordon model*, *Nucl. Phys.* **B 675** (2003) 584 [hep-th/0301042].

[42] A. LeClair and G. Sierra, *Renormalization group limit-cycles and field theories for elliptic S-matrices*, hep-th/0403178.

[43] B. Schroer, T.T. Truong and P. Weisz, *Towards an explicit construction of the sine-Gordon field theory*, *Phys. Lett.* **B 63** (1976) 422.

[44] M. Karowski, H.J. Thun, T.T. Truong and P.H. Weisz, *On the uniqueness of a purely elastic s matrix in* $(1+1)$*-dimensions*, *Phys. Lett.* **B 67** (1977) 321.

[45] A.B. Zamolodchikov, *Exact S matrix of quantum sine-Gordon solitons*, *JETP Lett.* **25** (1977) 468.

[46] M. Karowski, *The bootstrap program for 1+1 dimensional field theoretic models with soliton behavior*, in *Field theoretic methods in particle physics*, W. Rühl ed., Plenum, New York 1980, pp. 307–324, Presented at Kaiserslautern NATO Inst. 1979.

[47] C.-N. Yang, *Some exact results for the many body problems in one dimension with repulsive delta function interaction*, *Phys. Rev. Lett.* **19** (1967) 1312.

[48] R.J. Baxter, *One-dimensional anisotropic heisenberg chain*, *Ann. Phys. (NY)* **70** (1972) 323.

[49] D.I. Olive, *Unitarity and the evolution of discontinuities*, *Nuovo Cim.* **26** (1962) 73.

[50] J.L. Miramontes, *Hermitian analyticity versus real analyticity in two-dimensional factorised S-matrix theories*, *Phys. Lett.* **B 455** (1999) 231 [hep-th/9901145].

[51] R. Eden, P. Landshoff, D.I. Olive and J. Polkinghorne, *The analytic S-matrix*, Cambridge University Press, Cambridge 1966.

[52] V. Kurak and J.A. Swieca, *Anti-particles as bound states of particles in the factorized S matrix framework*, *Phys. Lett.* **B 82** (1979) 289.

[53] M. Karowski, *On the bound state problem in (1+1)-dimensional field theories*, *Nucl. Phys.* **B 153** (1979) 244.

[54] A.B. Zamolodchikov and A.B. Zamolodchikov, *Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models*, *Ann. Phys. (NY)* **120** (1979) 253.

[55] P.G.O. Freund, T.R. Klassen and E. Melzer, *S matrices for perturbations of certain conformal field theories*, *Phys. Lett.* **B 229** (1989) 243.

[56] H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, *Affine Toda field theory and exact S matrices*, *Nucl. Phys.* **B 338** (1990) 689.

[57] G.V. Dunne, I. Halliday and P. Suranyi, *Bosonization of parafermionic conformal field theories*, *Nucl. Phys.* **B 325** (1989) 526.

[58] G. Breit and E.P. Wigner, *Capture of slow neutrons*, *Phys. Rev.* **49** (1936) 519.

[59] A.B. Zamolodchikov, *Thermodynamic bethe ansatz in relativistic models. Scaling three state potts and Lee-Yang models*, *Nucl. Phys.* **B 342** (1990) 695.

[60] C. Ahn, *Thermodynamics and form-factors of supersymmetric integrable field theories*, *Nucl. Phys.* **B 422** (1994) 449 [hep-th/9306146].

[61] M. Moriconi and K. Schoutens, *Thermodynamic Bethe ansatz for $N = 1$ supersymmetric theories*, *Nucl. Phys.* **B 464** (1996) 472 [hep-th/9511008].